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An Introduction to

Infinitesimal Diffeomorphism Equations

Ray Brown

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An Introduction to Infinitesimal Diffeomorphism Equations

First Edition

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0.1 About this Book

Infinitesimal Diffeomorphism Equations (IDEs)is an entirely new mathematical discipline within the field of Nonlinear Dynamics. In contrast to global dynamics which seeks to predict the long-term asymptotic behavior of a system, IDE theory is primarily focused on local short-term dynamics. In particular, the prediction of the dynamics of complex transients is central to IDE theory. This is because short term dynamics are of great practical importance. Examples of complex transients of interest are the formation of tornados, the outcome of battles, the emergence of ebola, the formation of rogue waves, the onset of seizures and heart attacks. To this end, an objective of IDE theory will be to develop new methods of predicting complex transients, most notable of which are chaotic transients, and to be able to discern the potential for the emergence of complex transients from the *form* of the equations used to model a system, whether derived from data or formulated analytically.

This new field and this book are the product of many years of analysis of complexity, chaos and their transients. The present study originated at the Mitre Corporation with Dr.James Ellenbogen in the 1980s and continued with collaborations with Professor Leon Chua, Professor Morris Hirsch, Dr. Michael Shlesinger at ONR and Professor Walter Freeman at the University of California, Berkeley.

The driving force behind this multi-year study is the Hirsch Conjecture, which was a statement to the mathematical community in 1983 of the importance of developing a theory that could identify the presence of chaos in a system from the *algebraic form* of the equations that define the system. While a complete theory of IDEs and a resolution of the Hirsch Conjecture are yet to be achieved, the theory of IDEs provides a start toward the goals set forth here and does achieves many results.

In order to make this book available to the broadest possible audience (at least juniors in college) it is necessary to fill in background on the analysis of chaos and complexity. This somewhat of a digression will make the book more complete and accessible to the unfamiliar reader as well and will provide a historical record of how the theory was conceived.

A note on computer code provided in this book: Most illustrations are accompanied by an abridged version of the computer code used to generate them. Microsoft Visual Basic V6 Professional© (VB6) is the application used for all illustrations. VB6 was chosen for five reasons: (1) VB6 is a simple and efficient application for writing scientific code. Most programs can be written with only a few lines of VB code; (2) even a child can learn to use the minimal level of VB6 necessary to code quickly and explore the code without assistance; (3) VB6 provides a completely natural means of writing code which more closely aligns with high school algebra; (4) VB6 code, like its distant predecessor FORTRAN, parallels how models are developed; (5) VB6 is readily available and inexpensive.

0.2 Preface

Infinitesimal Diffeomorphism Equations (IDE) arose as an means of addressing the Hirsch Conjecture [81]. The Hirsch conjecture, briefly stated, asserts that it should possible to determine whether a differential equation has chaotic solutions by simply examining its form.

To address Hirsch's conjecture it is necessary to define what is meant by form and in this work, form will be defined as algebraic form. Additionally it is necessary to rigorously settle on what is meant by chaos. There are many definitions: (1) Exponentially sensitive dependence on initial conditions; (2) a positive Liaponov exponent; (3) a transverse homoclinic point, to name three. Of these, the most mathematically rigorously supported is the third. Its basis is known as the Smale-Birkhoff Theorem [130]. The theorem characterizes what is meant by *chaos* by proving that the level of complexity in a system having a transverse homoclinic point is, on an invariant subset, equivalent to a Bernoulli shift [140]. For another proof of the Smale-Birkhoff theorem see Nitecki, [110], page 154.

A Bernoulli shift is the mathematical idealization of a coin toss. However, the orbit of a shift is no more complicated than the initial condition of the orbit. This leads to the question of what is a complicated initial condition. Since the number that determines an initial condition may be considered to be a binary sequence, the question of how complicated an orbit is depends on how complicate a binary sequence is. Kolomogrov analyzed [64] how complicated a binary sequence may be and formulated a measure that can be rigorously applied. His metric is known as algorithmic complexity and the highest level of algorithmic complexity is *positive algorithmic complexity* formulated by Alekseev [4]. Since every binary sequence can be identified with a number between zero and one, another mathematician, per Martin Lof [64] proved that almost every number in the interval $[0, 1]$ has positive algorithmic complexity.

The research on complexity makes clear that the definition of chaos must be considered within the context of the more general notion of complexity since chaos and complexity are linked through the Bernoulli shift. This fact leads to a more generalized version of the Hirsch Conjecture that states that it should be possible to determine the level of complexity of a system from its algebraic form.

While there are various definitions of chaos, complexity is a more wide-ranging concept and is addressed indirectly in Ergodic theory [140] which provides the most organized approach to the level of complexity (a complexity spectrum) in dynamical systems. The spectrum runs from from ergodic to Bernoulli; and, Bernoulli systems are further classified by their entropy. However, this overlooks the practical matter that a non ergodic system may be temporarily complex. And this is what makes systems in the real world so difficult to predict. For example, periodic systems are thought of as simple. However, if the periodic system alternates between something as simple as the orbit of the hands of a clock from noon to midnight to the complexity of the three-body problem, the periodic system is difficult to predict. This example raises the question of what are the process in nature that drive a seemingly periodic system to alternate between simple dynamics and complex dynamics. In particular, while it may be possible to characterize the components of a periodic system, understanding of the transition between those components may remain elusive. Therefore, the *transition* between states of a real-world dynamical system (such as weather) is a part of the complexity of the system and must be mathematically addressed.

In addition to the issue of identifying complexity from the form of a model, there is also the issue of long term trends (global dynamics) and short term trends (local dynamics, esp. transients). The trends that are of the most immediate practical interest are short term trends. Predicting the formation of a tornado is the best example. Implicit in the prediction of short term dynamics is how systems transition from one level of complexity to another. The answer to this question is essential to predicting the formation of a tornado or the onset of an epileptic seizure. Conventional ODE theory and Newtonian dynamics are hard-pressed to address such questions common in the *natural* world. These considerations further enlarge the Hirsch Conjecture to state that it should be possible to determine both the long-term and short-terms levels of complexity of a system from the algebraic form of the model of a system. This is the context in which the Theory of Infinitesimal Diffeomorphism Equations arose.

IDE theory provides an avenue through which to organize questions of complexity, to examine the transitions between levels of complexity and to identify both long-term and short-term dynamics. This line of thought offers a generalization of the Hirsch Conjecture:

> *Is it possible to formulate a dynamical system so that its level of complexity, its transitions between levels and the short term and long term levels of complexity are revealed by its algebraic form ?*

Addressing the generalized Hirsch Conjecture is fundamental to IDE theory.

In order to address complexity more generally, there is a need for basic organizing concepts to be used in place of Newtonian dynamics and ODEs since Bernoulli shifts are not formulated in terms ODEs or Newtonian dynamics. This point leads to the choice of *stretching and folding* as basic organizing concepts due to their use by Smale [130] in understanding the formation of chaos in the three-body problem. Serendipitously, the dynamics that originates by combining stretching and folding lend themselves well to processes in nature and the social sciences where periodic processes (folding) combine with very complex stressful (stretching) evolutionary processes to drive the formation of life, weather, and geology. As a result, IDE theory makes extensive use of the concept of stretching and folding in place of Newtonian dynamics to formulate models. In particular, it is possible to use stretching and folding to formulate IDEs in a manner that makes the dynamics of complexity transparent from the algebraic form of IDEs.

> The ideal solution of the Hirsch Conjecture would be a theorem that states how any IDE can be decomposed into a complexity series consisting of elementary complexity building blocks that can be classified by their level of complexity by a single number such as entropy.
>
> Included in this representation would be the transitions between the building blocks as well as both the macroscopic and microscopic dynamics which represent the local and global dynamics.
>
> The series must be in closed form in terms of elementary functions with the provision that very insignificant terms can be discarded from any infinite expansions to achieve closed form solutions that are of practical value.

To address the Hirsch Conjecture, the form of an IDE must meet the following criteria:

1. Have a closed-form algebraic representation in terms of elementary functions
2. Explicitly reveal the dominate fixed points
3. Explicitly reveal the transitions between the dynamics determined by its dominate fixed points
4. Include both long-term and short-term dynamics

As a practical matter, IDE theory must show how to leverage other fields of mathematics to advance the theory. Also needed is how to use a *calculus* of IDEs to develop complex models in all sciences. And, when an IDE arises from and ODE, the IDE so derived should be a local solution of the ODE.

In summary, the Theory of Infinitesimal Diffeomorphism Equations addresses the question of complexity, i.e., the generalized Hirsch Conjecture; provides an alternative to ODEs; frees biological and social sciences from the burden of having to use Newton's laws by relying on the fundamental source of complexity as revealed by the Smale-Birkhoff theorem; provides an avenue for formulating problems in closed-form in terms of elementary functions; provides a calculus for building up complex IDEs from simple parts; provides a means for formulating models with the use of empirical data; and, provides a new method of *numerical analysis* and integration.

How does the study of IDEs assist and advance the work of other scientific enterprises? Infinitesimal diffeomorphisms (IDEs) [4] are transformations on a manifold that can closely approximate the solution of a differential equation. However, they are a legitimate subject of analysis in their own right due to (1) their potential application in the biological and social sciences as seen in [7]; (2) their use in the numerical approximation of the solutions of ODEs; (3) their use as closed form diffeomorphisms having complex dynamics that are equivalent to such systems as that of Chua, Lorenz and Rössler thus facilitating the direct study of such systems without the need of ODEs; (4) their independence from the laws of physics; (5) their use in modeling and simulation of large complex systems that presently require hundreds of ODEs to simulate and study; (6) their use in understanding the dynamics of complexity; (7) their use in constructing morphologically equivalent systems that can be expressed in closed form in terms of elementary functions. In this respect they provide morphological solutions of ODEs which cannot be solved in closed form in terms of elementary functions, or require conventional numerical methods to solve. For example, there is no closed form solution of the forced Duffing's equation in terms of elementary functions; however, there is an IDE solution in terms of elementary functions. (8) Statistical methods and even Stochastic Differential Equations only provide probabilistic correlations between dynamical parameters whereas IDEs provide cause and effect relationships between parameters.

The importance of IDEs to the study of the morphology of systems is made clear by the human EEG [2]: it is the morphology that determines normal versus clinical status of a human brain. Further, as is demonstrated in evolution, when chaotic systems and events unfold, they only rely on the occurrence of a frequency component rather than the order of occurrence of the frequency component in the dynamic of a phenomena or process. This is the morphology of natural systems. Morphology is nature's way of eliminating the importance of the specificity of the initial conditions in the origination of the dynamics of chaotic or complex process or events. For example, it is well-known that chaotic processes have sensitive dependence on initial condition while still having the same Fourier spectrum. This means that the exact initial conditions are not relevant so long as they are not too far apart, because all chaotic processes which start in a neighborhood of each other lead to the same morphological dynamic. This is the fact that biological and social dynamics depend on for their time evolution: some degree of independence from the initial conditions and the unfolding of the relevant components in any order, which may be random. The example of the tobacco mosaic virus provides a metaphor. If the virus is decomposed into its components and then place in a test tube, it can reassemble itself. Clearly, the order/arrangement in which the components appear in the liquid are not important, but only that they are present and available for a random process to facilitate the reassembly of the virus. IDEs provide very direct insight into the morphology of the dynamics of any system. (9) The "laws" on which social and biological systems depend to facilitate the formation of any degree of complexity are stretching and folding. IDEs are specifically formulated from these two dynamics and are thus ideally suited to study the morphological dynamics of complex systems.

To summarize, IDEs are formulated in terms of the fundamental source of *complexity* dynamics of biological and social systems rather than the laws of Newton; IDEs can be used to predict the dynamics of systems rather than describe the correlation between systems; IDEs provide significant computational compression over the use of ODEs for the formulation of complex biological theories; IDEs, for a large class of ODEs of interest to the biological and social sciences, provide very accurate approximations of the solutions of ODEs; IDEs are formulated in closed form in terms of elementary functions thus allowing for simplicity in modeling, simulation and programming. IDEs are iterations as opposed as functions of time and are thus a form of numerical integration; IDEs are local solutions of ODEs when they arise from an ODE; when the there is an attractor component to the IDE, the IDE may be a global solution of an ODE; IDEs can be used to determine the long-term asymptotic dynamics of a system as is done in the subject of Global Analysis.